The Journal of Mind and Behavior
Spring and Summer 1997, Volume 18, Numbers 2 and 3
Pages 103 [1] – 126 [24]
ISSN 0271–0137
ISBN 0–930195–08–6

Chaos and Related Things: A Tutorial

Bruce J. West

University of North Texas

Chaos theory and related things are described by way of differences between traditional (linear) science and non-traditional (nonlinear) science. Differences described between linear and nonlinear models of science respectively include the following: quantitative vs qualitative, analytic vs non-analytic, predictability vs unpredictability, fundamental scaling vs scaling relations, and superposition vs emergence. Common themes in non-traditional science are the existence of nonlinearity, scaling relations, and unpredictability. Data are provided that show that many social and psychological phenomena can be understood only through nonlinear modeling. It is concluded that as the old and new views of science coalesce, the newer mathematical tools will help make understandable the irregular and erratic features of everyday life.

> Science is said to proceed on two legs, one of theory (or, loosely, of deduction) and the other of observation and experiment (or induction). Its progress, however, is less often a commanding stride than a kind of halting stagger — more like the path of the wandering minstrel than the straight-ruled trajectory of a military marching band.
>
> Timothy Ferris
> *Coming of Age in the Milky May*, 1988

A tutorial on the application of chaos and scaling to social and biomedical phenomena can take any number of forms. At one extreme would be a turgid collection of discrete and continuous equations bracketing definitions and proofs on page after lifeless page. The other extreme would be a superficial popularization emphasizing the most familiar terms such as fractals and

I would like to thank the office of Naval Research for partial support of the research on which this paper is based, with a special thanks to Walter Freeman for reading an early version of the paper and making a number of useful suggestions. Requests for reprints should be sent to Bruce J. West, Ph.D., Director, Center for Nonlinear Science, University of North Texas, P.O. Box 305370, Denton, Texas 76203.

chaos. Both of these "primers" serve a purpose, but my understanding is that neither would be satisfactory in the present context. Therefore we are going to take the high road and avoid excessive mathematical detail by not writing ideas in mathematical notation, but instead explaining important mathematical concepts in words. Of course it will not be possible to avoid equations altogether, and I appeal to the reader to try and meet me halfway in analyzing these equations. I am confident that the reward will be more than worth the effort.

Our approach to introducing the idea of chaos into the study of the social and psychological sciences shall be to examine the underlying assumptions of the physical sciences. This backdoor approach is taken because the physical sciences in general, and physics in particular, form the paradigm for the social and psychological sciences. Thus if we understand the weaknesses of the paradigm, then the arguments traditionally used to develop mathematical models in the latter case may not be so compelling. The basis of physics is the traditional linear perspective of Newton and his lineage, and we propose to discuss the challenges to that foundation made by the nonlinear perspective of Poincaré and his intellectual progeny. We follow a procedure that has had some success in the past and collect the conventional wisdom of the physical sciences into an assortment of traditional truths which over time have become so obvious that they are all but impossible to call into question. Thus to determine which strategies would be the most effective in the further development of theories of complex phenomena requires that we take cognizance of these mostly unconscious truths. In Table 1 we list five traditional truths, not necessarily in the order of their importance, along with a counterpoising set of five non-traditional truths which by and large seem to erode the foundations of the physical sciences and in so doing forces us to rethink the basis of theory in the social and psychological sciences. Here we are not interested in specific social models except insofar as they provide a context in which to understand large data sets.

The importance of chaos in the understanding of complex phenomena becomes apparent in the discussion of the changes in our scientific perspective. Chaos theory is used here as code for the rather intimidating phrase, *low-dimensional, deterministic, nonlinear, dynamical systems theory* and unquestionably frees the social and life scientist from the straight jacket of nineteenth century physics, without abandoning such basic concepts as determinism. On the other hand chaos broadens our concerns from the smooth structures and parallel lines of Euclidean geometry to the scale-free, self-similar forms of fractal geometry. Also rather than the self-serving predictions of traditional theory, we offer a mathematics to describe the counter-intuitive, unpredictable evolution of complex systems. It is the property of chaos to be just as crazy as complex phenomena in the social and psychological sciences that

Table 1
Traditional and Non-traditional Truths of Science

Traditional Truths	*Non-Traditional Truths*
1. Physical theories are and should be quantitative.	1. Qualitative theories are as important, and sometimes more important than quantitative ones.
2. Natural phenomena can by and large be represented by analytic functions.	2. Many phenomena are singular in character and cannot be represented by analytic functions (fractals).
3. Natural phenomena have fundamental scales.	3. Natural phenomena do not necessarily have fundamental scales and may be described by scaling relations.
4. The evolution of natural phenomena can be predicted from the equations of motion.	4. The evolution of many phenomena, although derivable from dynamical equations, are not necessarily predictable (chaos).
5. Most phenomena satisfy the principle of superposition.	5. Most phenomena violate the principle of superposition (emergence).

make it such an attractive descriptor. That is to say that the scale-free, fractal nature of nonlinear deterministic dynamics (chaos) often leads to unexpected and sometimes unacceptable changes in time series thereby mimicking similar complex dynamics in the social and natural sciences, for example, sudden changes in a person's behavior related to a small, seemingly unimportant incident.

Therefore a large part of our attention in this tutorial shall be on data sets that demonstrate the scaling nature of social and psychological phenomena and discussions of how these data challenge the traditional paradigm. We show both the continuous and discrete logistic equation, the former having a significant pedigree in the social sciences, whereas the latter has a remarkable, if shorter, history in the mathematical sciences. The continuous logistic equation is shown to describe the spreading of information and innovations in society, and to result in regular saturation phenomena. The discrete logistic equation on the other hand is seen to result in regular, periodic and scale-free chaotic phenomena. The discrete dynamical equations in general are infinitely richer in the type of time-dependence they can and do describe.

We emphasize that everything we shall be discussing herein is based on the idea that all interesting natural and psychological phenomena are fundamentally not linear, or said differently, all interesting phenomena are nonlinear. This cannot be overstated because most concepts in science are either implicitly or explicitly linear in character. It is the assumption of linearity that underlies the traditional truths contained in Table 1, and it is the existence of nonlinearities that undercuts their truth value. One property of linearity that is rarely, if ever, observed in biomedical social or psychological phenomena is that of proportionality: the response of a system is proportional to the stimulus. A nonlinear system, on the other hand, is one for which the size of the input does not determine the size of the output. Such fundamental concepts as threshold and saturation are intrinsically nonlinear and cannot be approximated by linear functions. This is the case even when a transformation of the data makes it amenable to a linear description, such as the common method of taking logarithms of the data. The transformation itself does not make the process linear nor does it give any additional insight into actual system dynamics. For a linear system containing many factors the total response of the system is proportional to the sum of the individual responses of each of the separate factors. This is the property of independence. For a linear system there always exists a representation in which the factors are mutually independent. However, this is not true in general and the interdependence of the factors is a manifestation of nonlinearity. Finally it should be emphasized that the notion of an *emergent* property in which "the whole is greater than the sum of its parts" contradicts traditional linear theory. It is only with nonlinear phenomena that a property not explicit in the underlying elements can emerge through the interaction of these elements. The total response of a nonlinear system is always a complex function of the input, and for a chaotic system this is a very sensitive function. *By sensitive we mean that a minuscule change in the input can have a catastrophic change in the output.*

We emphasize at the outset that we draw no conclusions herein regarding particular models, nor do we develop a theory of nonlinear social or psychological interactions. So what do we have to say that might be of value? Simply that the strategies for modeling complex social and psychological phenomena that have been used in the past, insofar as they either implicitly or explicitly rely on linear assumptions, are overly restrictive, and chaos may well provided a fresh, new approach that is more faithful to reality.

Truth or Consequences

We have separated what we consider to be the underlying assumptions of the physical sciences into five traditional truths. We may debate about the independence of these truths and whether or not the list is complete, but let us for the sake of argument tentatively accept them. We believe the list is a reasonable one and what is significant about it for us here is how the nonlinear interactions in the dynamics of natural and social phenomena call each and every item on the list into question. A similar exercise could be carried out with any other such list with similar results. An appreciation of how nonlinear concepts in dynamics undermine our cherished preconceptions takes us a long way down the road to understanding complex phenomena, the time series we measure to represent them and their statistics. Much of our failure to understand is more a matter of misunderstanding due to the inapplicability of unspoken assumptions than it is a matter of not being able to conceptualize what is true about a phenomenon. This is in part the reason we examine the traditional method of modeling simple social and biomedical phenomena: to be able to point out the apparent inconsistencies with the methods used to model complex natural phenomena.

Qualitative Versus Quantitative Science

The first and perhaps foremost of the traditional truths is that scientific theories are (and should be) quantitative. As Lord Rutherford phrased it: "All science is either physics or stamp collecting." The viewpoint was clarified by René Thom (1975) who pointed out that by the end of the seventeenth century there were two main groups in physics, those that followed the physics of Descartes and those that followed the dictates of Newton:

> Descartes, with his vortices, his hooked atoms, and the like, explained everything and calculated nothing. Newton, with the inverse square law of gravitation, calculated everything and explained nothing. History has endorsed Newton and relegated the Cartesian constructions to the domain of curious speculation. (p. 763)

Thus this first traditional truth takes the alternate form that *if it is not quantitative it is not scientific*. This visceral belief has molded the science of the twentieth century, in particular those emerging disciplines relating to life and society have by and large accepted the need for quantitative measures. It is not that this perspective is wrong, but rather that it is overly restrictive. It does not enable the psychologist to understand the aesthetic judgment we make in viewing a painting or listening to music; nor does it assist the politician in reaching a decision concerning the quality of life of his/her constituency. For these latter considerations we need to examine the qualitative as well as the quantitative aspects of the world. For this reason the first of the non-traditional truths is that scientific theories should be qualitative as well as quantitative. The most venerable proponent of this view in recent times was D'Arcy Thompson (1917/1963), whose work in part motivated the development of catastrophe theory by Thom (1975). Their interest in biological morphogenesis stimulated a new way of thinking about change — not the smooth, continuous quantitative change familiar in many physical phenomena, but the abrupt, discontinuous, qualitative change familiar from the experience of "getting a joke," "having an insight," or the bursting of a bubble.

Many if not most interesting phenomena in nature and society involve discontinuities, sudden changes, and it is only relatively recently in the history of science that we have available the appropriate mathematical disciplines to describe the behavior of such changes. Catastrophe theory and topology are two examples of the approaches that stand in sharp contrast to the vast majority of available techniques which were developed in the physical sciences for the quantitative study of continuous behavior. Catastrophe theory has a mathematically rigorous foundation in topology and is qualitative rather than quantitative in nature, which is to say it deals with the forms of things and not with their magnitudes. In topology an orange and the earth are the same insofar as they are both spheres and can only be distinguished by their radii. Further, all shapes that can be achieved by smoothly deforming a sphere are topologically indistinguishable. Thus a sphere and a bowl are topologically the same, but a cup with a handle is different since the handle has a hole in it. However, a cup and a doughnut, or any other one-holed shape are topologically equivalent. Here the no-hole, one-hole, two-hole, and so forth aspect of things determines their qualitative nature. This theory recognizes that many, if not most, interesting phenomena in nature and society involve discontinuities. An example of a possible missed application of this theory was anticipating the fall of the Berlin Wall. To my knowledge no political scientist foresaw this abrupt change in the world politique, but it was to describe just this kind of sudden bifurcation that catastrophe theory was developed. Catastrophes are abrupt changes arising as a sudden response of a system to a smooth change in external conditions. It is more general than the term bifurcation in that the latter is a forking of various entities resulting from changes in the parameters on which a system depends. Unfortunately when catastrophe theory was first introduced a number of its more zealous supporters made unrealistic claims for what it could accomplish and when these claims were not realized its subsequent reputation was tarnished. Among the various applications of catastrophe theory that were of limited or no success were models of the activity of the brain and mental disorders, prison uprisings, the behavior of investors on the stock exchange, and the influence of alcohol on drivers (see, for example, Arnold [1986]).

Topology is but one example of a rigorous mathematical discipline whose applications to the physical, biological and social sciences emphasize the qualitative over the quantitative. Another is the bifurcation behavior of deterministic nonlinear dynamical equations. A bifurcation is a qualitative change in the solution to a differential or difference equation obtained through the variation of a control parameter. For example, certain systems manifest periodic motion, a behavior that repeats itself after a fixed interval of time, such as the sun coming up each morning, changing a control parameter: the process generates a sequence of sub harmonic bifurcation's in which the period of the motion doubles, doubles again and again and so on (see, for example, West and Deering [1995]). Eventually the motion, after becoming more and more complex with each bifurcation, winds up being irregular in time, as we discuss below. This irregular behavior has been used to model such apparently random phenomena as the transfer of disease in a population during an epidemic, oscillating chemical reactions and the clear air turbulence experienced by airline travelers. The periodic behavior of the final states has suggested a new paradigm for the unpredictable behavior of complex systems, the phenomenon of chaos (see West [1990] for a review of the applications of these ideas to biomedical phenomena and Ott [1993] for a rigorous mathematical treatment). Of course we have only mentioned one of the many possible routes to chaos in the above example, that being the sub harmonic bifurcation route. Another is an intermittency transition to chaos, where a system is regular for a parameter below a critical value. Above this value the system has long time intervals in which its behavior is periodic (regular), but this apparently regular behavior is intermittently interrupted by a finite-duration burst during which the dynamics are qualitatively different. The time intervals between bursts are apparently random. As the parameter value becomes significantly larger than the critical value the bursts become more and more frequent, until finally only the bursts remain. There are three types of intermittency transitions. One is called a Hopf-bifurcation and was used by Freeman (1994) to describe the onset of low-dimensional chaos in the brain. These arguments suggest the form of the first non-traditional truth: *qualitative descriptions can be as important, if not more important, than quantitative descriptions in science*.

Analytic Functions and Fractals

The second fundamental truth spans the macroscopic and microscopic worlds of physical science and maintains that physical observables and their relationships are represented by analytic functions. Since the time of Lagrange (1759) it has been accepted that celestial mechanics and physics are described by smooth, continuous, and unique functions. This belief is part of the infrastructure of the physical sciences because the evolution of physical processes are modeled by systems of dynamical equations and the solutions to such equations are thought to be continuous and differentiable on all but a finite number of points, that is to say, the solutions are analytic functions. We see here one of the major assumptions of the physical sciences, that being that understanding comes from prediction and description, not from the identification of teleological causation. Thus, science is concerned with the *how* of phenomena not with the *why*. Galileo, condensed the activity of science into the following three tenets:

(1) description is the pursuit of science, not causation;
(2) science should follow mathematical, that is, deductive reasoning;
(3) first principles come from experiments, not the intellect.

This summary of Galileo's scientific philosophy set the stage for Newton who was born the year Galileo died, 1642. Newton embraced the philosophy of Galileo and in so doing inferred mathematical premises from experiments rather than from physical hypothesis. This approach of proceeding from the data to the model is no less viable today than it was over three hundred years ago and we follow it herein.

According to the above, the purpose of modeling is to obtain a simple description that captures the essential features of the process being investigated. In physics one recalls in the static case such expressions as Ohm's law $E=IR$ where E is the voltage measured across a resistance R through which a current I is flowing, and the perfect gas law $PV=NkT$ where P is the pressure of a gas contained in a volume V at a temperature T, and N is the number of particles with k being a known constant. These are the kinds of expressions sought in other areas of science as well. But such simple laws are usually not available outside the physical sciences. On the other hand, one does encounter relations of the form $Y=\alpha M^{\beta}$, allometric growth laws, that appear in biology, botany, sociology and other branches of science. The allometric equations in physiology, say, associate Y with any physiological, morphological or ecological variable and M, in most cases, is the body mass and α and β are constants determined by the data. The exponent β is usually not a rational number, which is to say that it cannot be written as the ratio of integers, see for example, Calder (1984) and/or MacDonald (1983). Herein we put these power laws into a somewhat broader context.

In 1738, Daniel Bernoulli expressed the significance of inverse power laws, or $1/f$-phenomena, when he devised a quantity called the utility function. He was interested in characterizing the social behavior of individuals, and the utility function was intended to describe an individual's social well-being. He reasoned that a change in some unspecified quantity f, denoted by Δf, has different meanings to different persons depending on how much they already possess. For example if f denotes the level of a person's wealth then the greater f the less important is any particular incremental change in that wealth Δf. Suppose you and I both invest money in the stock market, you invest \$100 and I invest \$10. If we both make \$10 profit, who is the happier? You that made 10% profit or me that made 100% profit? The utility function specifies that individuals respond to percentage changes, not to absolute changes, so the larger response corresponds to the larger $\Delta f/f$. In other words, I am happier than you even though we both made the same amount of money, because my percentage change was greater than yours. We shall have reason to use this observation repeatedly.

Although such a simple model as the utility function does not provide a complete characterization of an individual's response, it does seem to capture an essential aspect of that response, that being its scale-free nature. Everything we see, smell, taste, and otherwise experience is in a continual process of change. But the changes in the world are not experienced linearly, which is to say that our responses to the changes are not in direct proportion to the change. In the last century the physiologist E.H. Weber studied the sensations of sound and touch and experimentally determined that people do not respond to the absolute level of stimulation but rather to the percentage change in stimulation. Shortly thereafter the physicist Gustav Fechner founded a new school of experimental psychology called psychophysics. He determined the domain of validity of Weber's findings and renamed it the Weber–Fechner law (Fechner, 1860). If the change in frequency Δf is the stimulation, then we respond to $\Delta f/f$ rather than to Δf itself. This early work supported the intuition of Bernoulli even though its basis was biological rather than social.

Stevens (1957) argued that the logarithm law of Weber and Fechner should be replaced by the power law

$$Y(X) = \alpha X^{\beta} \quad (1)$$

for constants α and β with $\alpha > 0$; X is the intensity of the applied stimulus and $Y(X)$ is the perceived (nonlinear) response. This idea dates back to Plateau in 1872, and the equation has the same form as the allometric growth law. Consider the subjective loudness of a sound relative to a reference sound. Suppose the reference sound is given a number and a sound perceived as half as loud is given half the value of the reference number. The graph of a sequence of these numbers versus the actual intensity of the sound is shown in Figure 1. The straight line on this log–log graph clearly indicates a power law of the form (equation 1), which is to say that a logarithmic transformation of the data makes the response linear in the new variable. All sensory systems seem to perform this transformation to achieve range compression. It should be noted however, that the extremes of the sigmoidal shape seen in the data, the threshold and the saturation, cannot be linearized. A number of stimuli, including brightness, taste, smell, temperature, vibration, duration, pressure, pain, and electrical shock to name just a few have all been used to determine the empirical values of α and β for different modalities (see Roberts [1979] for a complete review of theory and experiments). It is not only in psychophysics that these laws exist, however. They appear in biology, economics, sociology as well as in other areas of study.

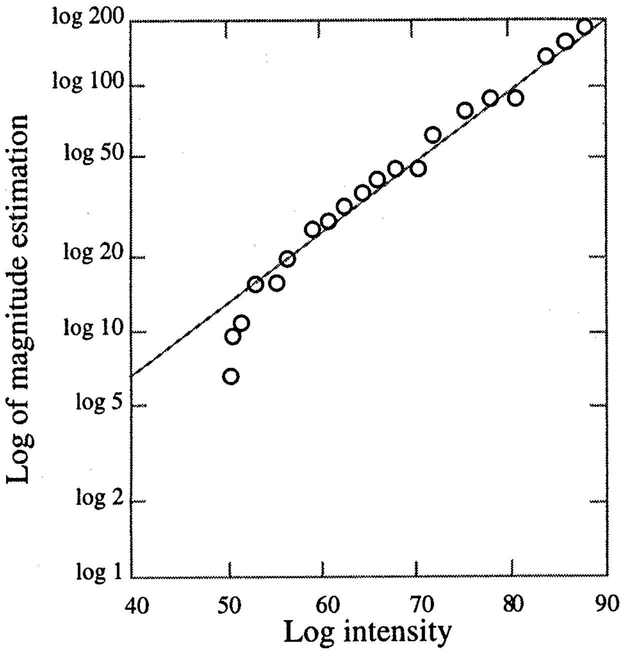

Figure 1: Magnitude estimation judgments in log–log coordinates, fitted with a straight line of slope 0.3. From "The Relation between Category and Magnitude Scales of Loudness," by E. Galanter and S. Messick, 1961, *Psychological Review*, 68, p. 366. © 1961, American Psychological Association. Adapted with permission.

F. Auerbach published an empirical law of urban concentration in 1923. He found that the product of a city's population and its rank in a given country produce a constant. If the cities in a country are ordered from largest to smallest in population, a city's rank is its ordinal position in the sequence. In this way he obtained a hyperbolic curve by plotting population versus rank, or more dramatically, he obtained a straight line by plotting the logarithm of one quantity versus the logarithm of the other. Auerbach's Law has the form NR^{α}=constant, with α=1 here N is the population of the city and R is its rank. Note that this equation is of the form (equation 1) with a negative exponent. If one plots the data for the 100 largest cities in the United States from around 1920 we obtain α=0.93 rather than α=1.

Our earlier example of individual response had to do with our reaction to the change in our wealth. One of the earliest quantitative investigations of the distribution wealth was made by the engineer turned sociologist, Vilfredo Pareto, published 1897. He collected statistics on the income and wealth of individuals in many western countries at various times in history. He sought to understand how wealth was distributed in western societies and from this understanding to determine how such societies operate. In Figure 2 we depict the distribution of income in the United States in 1918 on log–log graph paper. A straight line with a negative slope indicates an inverse power law, with the index (being given by the slope of the line, see equation 1). This is the so-called Pareto's Law of income distribution. Pareto believed that α had the universal value of 1.5 for western societies, but this turned out not to be the case. If we normalize the data in this figure differently, for example, dividing by the total population, then we can interpret this curve as a probability distribution function, since we have a relative frequency for the number of people in a given income interval. The curve abruptly stops at

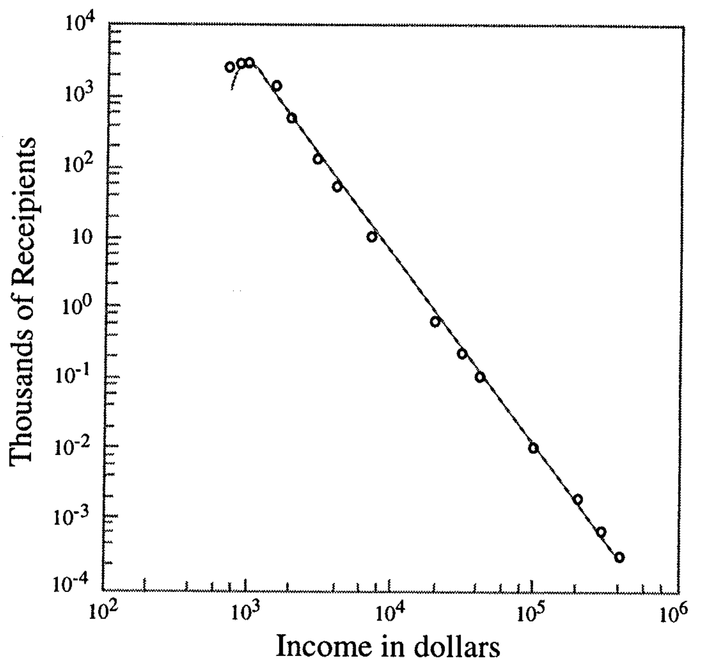

Figure 2: *Pareto's Law:* Frequency distribution of incomes in the United States, 1918. The value of 1.5 for the Pareto index is not as universal as he had hoped.

$600, the so-called "wolf point," that being the point at which the wolf is at the door. In 1918 the income of $600/year was the minimum income level necessary to maintain life with a modicum of dignity. These inverse power laws are the result of multiplicative amplification processes that have been called *the principle of statistical leverage*. This means that people in the highest income domain operate in a way different from most individuals in the income distribution. While most people are paid an hourly wage, the wealthy frequently accumulate their extra wealth by means of some amplification process: that process varying from case to case. Perhaps one of the most common lower level modes of amplification is for an individual to organize an operation with others working for him/her so that his/her income is amplified through the efforts of others (a modest sized business, for example) [see Montroll and Badger (1974) or West and Deering (1995) for a more complete discussion]. Income distributions such as this one were found for many western societies by Pareto as well as subsequent investigators, independently of their particular social organizing principles.

In a quite different social context Lotka (1926) observed another inverse power-law distribution. Lotka was interested in the number of papers published by scientists in a given year. If the fraction of the total number of scientists versus the number of papers published, in the order of the least frequent to most frequent, is graphed on log–log graph paper, the distribution is an inverse power law in the number of papers published and α in (1) is approximately 2, compare Figure 3. From this figure we see that for every 100 scientists who publish a single paper in a given period of time, there are 25 scientists who publish two, 11 with three, and so on. Putting this in cumulative form, we find there is one in five scientists who produce five papers or more; one in ten who produce at least ten papers, and so on. It is interesting to note that although there is no guarantee that a scientist who only publishes a few papers will not achieve international recognition, or that a scientist who publishes a great deal will achieve prominence, there is in fact a strong correlation between the number of publications and a scientist's reputation (de Sola Price, 1963). Note that the number of papers published has been leveraged in a way completely equivalent to that used for income. The publication of scientific papers is leveraged through the use of graduate students and postdoctoral researchers under the direction of one or more senior scientists. The latter receive the advantage of co-authorship on multiple publications; more than they could have produced working alone. The coworkers likewise benefit from the guidance of the seasoned veteran.

As a final example of these hyperbolic (inverse power law) relations we draw from the arena of biological evolution and the work of Willis (1922). Willis argued that the spatial area occupied by a biological species is directly proportional to the age of the species, so the area is a measure of age.

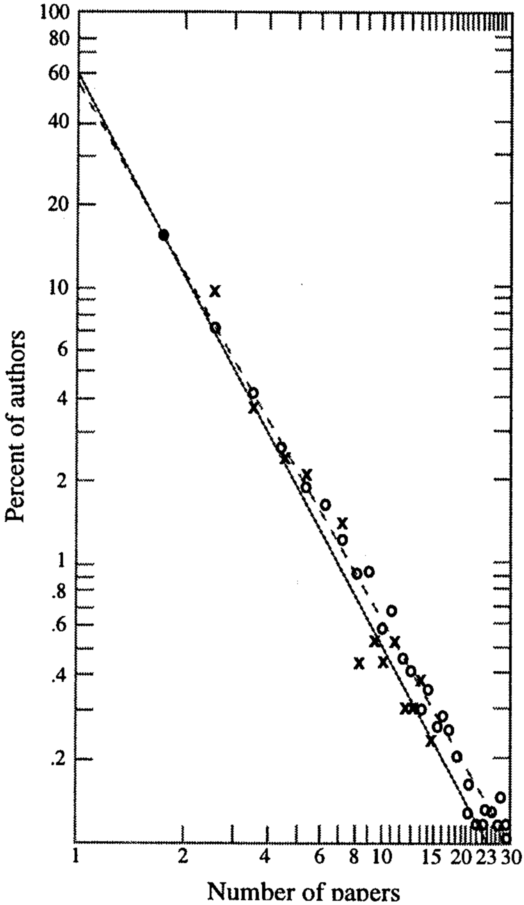

Figure 3: *Lotka's Law:* The number of authors publishing exactly n papers, as a function of n. The open circles represent data taken from the first index volume of the abridged *Philosophical Transactions of the Royal Society of London* (seventeenth and early eighteenth centuries), the filled circles those from the 1907–16 decennial index of *Chemical Abstracts*. The straight line shows the exact inverse-square law of Lotka. All data are reduced to a basis of exactly 100 authors publishing but a single paper. From "The Frequency Distribution of Scientific Activity," by A.J. Lotka, 1926, *Journal of the Washington Academy of Science*, *16*, p. 317. © 1926, Washington Academy of Sciences. Reprinted with permission.

Leaving the details of his theory aside, Willis collected data on various natural families of plants and animals and graphed the numbers of genera as ordinate and the number of species in each genus as abscissa. The relation between the number and size of genera of all flowering plants is a hyperbolic distribution, see for example Figure 4. The monotypic genera, with one species each, are always the most numerous; the ditypics, with two species each, are next in rank, genera with higher numbers of species becoming successively fewer. The relation is most apparent when displayed on log–log graph paper, where we obtain a straight line with a slope $-\beta$. The inverse power law was found to be true of all flowering plants, certain families of beetles as well as for other groupings of plants and animals. The basis for Willis' Law was argued to be the result of mutations (genetic variation), and we shall support this argument somewhat later in a more general context when we determine the possible underlying reason for inverse power laws in all these complex phenomena.

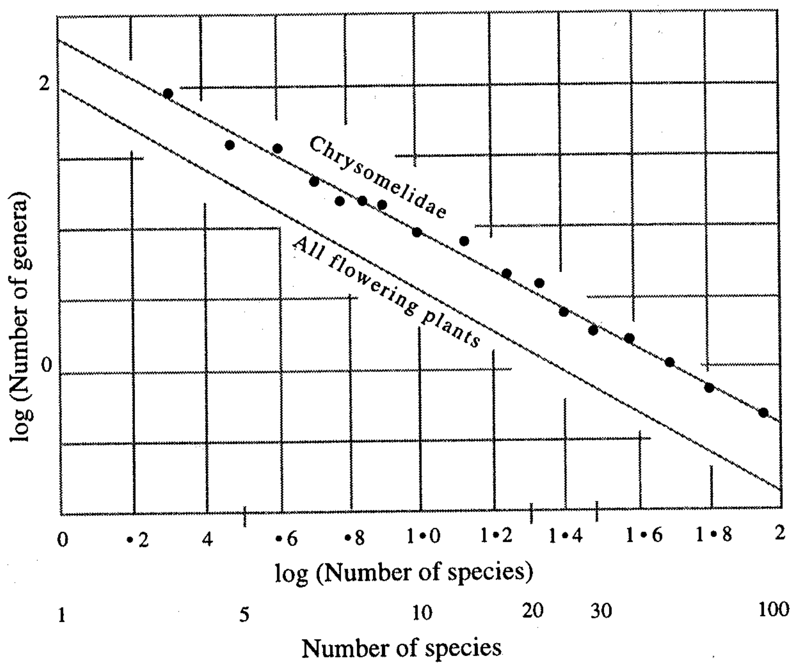

Figure 4: *Willis' Law:* Relation between number and size of genera of chrysomelid beetles compared with all flowering plants. From *Age and Area: A Study in Geographical Distribution and Origin of Species* (p. 242), by J.C. Willis. © 1922, Cambridge University Press, Cambridge, England. Reprinted with permission.

Recall that each of the above inverse power laws is given as an example of a complex social or biological phenomena for which the simple analytic functions of the physical sciences are inadequate, but which can be linearized by logarithmic transformation. However, there has always been a substantial number of natural phenomena that could not be linearized by logarithmic or any other transformation or described by analytic functions. The brain is not a sphere, trees are not cones, the transition of water to ice does not happen continuously, hiccupping and shivering are not smooth dynamical processes, and on and on. In phenomena such as these, scaling is crucial in the description of the underlying process. In fact the inverse power law is a direct consequence of scaling. The existence of scaling relations, implies that complex systems cannot be described by analytic functions. Modern scaling arguments result from the fact that these systems do not have a fundamental scale.

From this discussion about scales we see the importance of the form of the equations of motion used to describe the time evolution of the system of interest. Simple analytic functions are the solutions to differential equations with well-behaved boundary and/or initial conditions. The smooth, continuous behavior of the solutions seems to be anticipated by the forms of the equations of motion. It has been recognized that the time dependence of many discontinuous phenomena as well as the structure of many fractal geometrical forms are determined by recursive algorithms rather than by differential equations. We shall find that discrete equations, also called mappings, are a very powerful strategy for modeling complex processes that cannot be represented by analytic functions, but are fractal processes that do not possess a characteristic scale. Thus the second non-traditional truth is: *many phenomena are singular in character and cannot be represented by analytic functions*.

Scaling Relations

The third traditional truth is that physical systems can be characterized by fundamental scales such as those of length and time. Such scales provide the meaning of the fundamental units in the physical sciences without which measurements could not be made and quantification would not be possible. On the other hand the new ideas introduced above require that we reinterpret existing data sets. For example, if we consider an irregular time series our first impression is the lack of organization associated with the random fluctuations. If we focus our attention on a small interval and magnify the fluctuations, there are two kinds of results that we might encounter. The first kind of result is that in the magnified region of the time series the curve becomes smooth and no more irregularity is observed. That is what happens if the time series can be represented with an analytic function. There is always a smallest scale below which there is no further variability. The second kind of result is that no matter how much we magnify the curve, more and more structure is uncovered. What is observed on one scale is repeated again and again on adjacent scales, cascading upward and downward, never becoming smooth, always revealing more and more structure.

Thus we have wiggles within wiggles within wiggles. Mathematically, there is no limiting smallest scale size and therefore there is no scale at which the variations in the function (data) subsides. Of course in the real world there is always a smallest scale, however if this cascade covers sufficiently many decades of scale it is useful to treat the natural function as if it shared all the properties of the mathematical function including the lack of a fundamental scale. This idea came into sharp focus using real data in the classic work of Lewis Fry Richardson.

Richardson was a Quaker and an ambulance driver in "the war to end all wars." He was also quite interested in the reasons for the causes of wars. In this pursuit, since border disputes are often used as a reason for conflict, he became interested in how we determine the borders between countries and the length of coastlines. He estimated the length of a coastline (border) by using a pair of calipers opened a fixed amount and then "walking" them along a border in an atlas. He found that the measured length of the border changed as the opening of the calipers became smaller. If r is the length of the ruler (caliper opening) used and $L(r)$ the length of the coastline, Richardson (1961) obtained the estimated length of an irregular coastline to be

$$L(r) = L_0 r^{1-D} \qquad (2)$$

where L_0, is the measured length of the coastline when the ruler is of unit length. The constant D is calculated from the slope of the straight line on a log–log plot of the data: $\log L(r) = \log L_0 + (1-D)\log r$. For a classical smooth line the dimension D equals one, and $L(r)$ becomes a constant independent of r, that is, the coefficient of the size of the ruler vanishes when $D=1$. For an irregular coastline it was found that $D>1$, the data for the total length of coastlines and boundaries fall on straight lines with slopes giving non-integer dimensions. From these data it is found, for example, that D is approximately 1.6 for the coastline of Britain and 1.0 for a circle, as expected (see Figure 5). It is also found that the length of a curve described by equation 2 diverges; it becomes infinitely long as the size of the ruler goes to zero for such irregular curves, since $(1-D)<0$. Such a curve was called a *fractal* by Mandelbrot (1977).

From equation 2 we observe that if we scale the ruler size by a constant λ, then we obtain $L(\lambda r) = \gamma^{\beta} L(r)$, where of course $\beta = 1-D$. This scaling relation is also observed in each of the examples given above: cities, wealth, biology, and so on. The scaling observed in fractals is seen in all the above power laws and inverse power-law distributions. It is precisely this scaling behavior that is manifest by the dimension, D, being non-integer in equation 2, or in equation 1 for that matter. Classical scaling principles are based on the notion that the underlying process is uniform, filling an interval in a smooth, continuous fashion. The new principle is one that can generate richly detailed, heterogeneous, but self-similar (or self-affine) structure on all scales. Thus

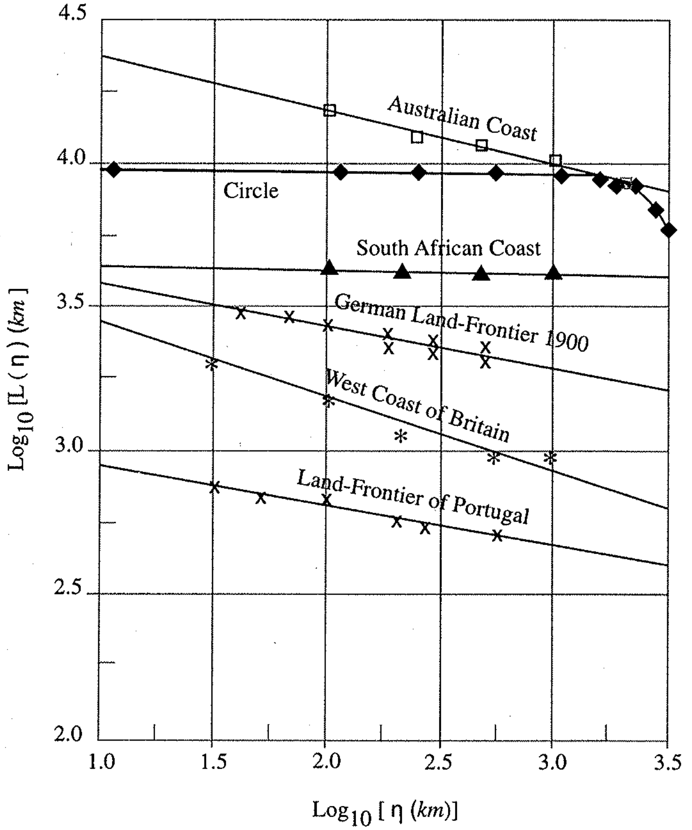

Figure 5: *Richardson's Law:* The logarithm of the length of coastlines as a function of the logarithm of the yardstick length shows an inverse power-law behavior. From "The Problem of Contiguity: An Appendix to Statistics of Deadly Quarrels," by L.F. Richardson, 1961, *General Systems Yearbook*, 6, p. 169. © 1961, International Society of Systems Sciences. Reprinted with permission.

the structure of many systems are determined by the scale of the measuring instrument, and such things as the length of a curve are a function of the unit of measure, for example, the length of a fractal curve depends on how it is measured. The third non-traditional truth is therefore that *natural phenomena do not necessarily have fundamental scales and may be described by scaling relations*.

Nonlinear Deterministic Equations and Chaos

The fourth traditional truth is that since the time evolution of physical systems is determined by systems of deterministic dynamical equations, if one completely specifies the initial state of the system then the solution to the dynamic equations uniquely determines the final state of that physical system. Thus the final state can be predicted from a given initial state using the dynamical equations. Until the last decade or so this view prompted most scientists to casually assert that the evolution of these observables are absolutely predictable. While it is true that the Newtonian model is valid, it does not imply absolute predictability for arbitrarily long times. The particular results with which we are concerned are the periodic (irregular) solutions mentioned earlier. These solutions are manifestations of the deterministic randomness (chaos) that can arise from intrinsic nonlinear interactions in the dynamical equations [see, for example, Crutchfield, Farmer, Packard, and Shaw (1986)]. Since chaotic processes are irregular, they have limited predictability and call into question each of the first four traditional truths, that is, whether the important aspects of complex phenomena are that they be quantitative, analytic, possess fundamental scales and are predictable.

Here we examine how a deterministic mechanism, for example, a set of deterministic rate equations, can give rise to erratic time series that satisfy all the conditions of randomness. We argue that there exists a deep relation between the idea of a fractal dimension and chaos. Chaotic time series are found to have a fractal dimension in precisely the same way an inverse power-law spectrum for a random time series has a fractal dimension. For this reason it is often not clear whether or not a given random fractal time series is generated by a low-dimensional deterministic nonlinear dynamical process or by colored noise; the latter being noise (a random time series arising from the interaction of the system of interest with the infinite environment) with an inverse power-law spectrum. This particular difficulty is beyond the scope of the present discussion and arises in the practical context of data analysis (Bassingthwaighte, Liebovitch, and West, 1994). That being said, it is quite important that we distinguish between chaos and noise in a given experimental time series, because how we subsequently analyze the data and interpret the underlying process are determined by this judgment. A colored noise signal generally implies that we look for a static fractal structure that is modulating the noisy signal in such a way as to give rise to the fractal dimension. If, however, the signal is chaotic, then the fractal dimension is related to the underlying nonlinear dynamical process and we have some hope of constructing a dynamical description of that underlying process.

The phase space for a dynamical system consists of coordinate axes defined by the independent variables for the system. Engineers refer to this as the "state space," owing to the use of the term "phase" to denote shifts in time between oscillators of the same frequency such as in brain wave (EEG) data. Each point in the phase space corresponds to a particular set of values of the dynamical variables that uniquely define the state of the system. This point moves about in phase space as the system evolves and leaves a trail that is indexed by the time. This trail is referred to as the orbit or trajectory of the system. In general a phase space orbit completely describes the evolution of a system through time. Each choice of an initial state produces a different trajectory. If however there is a limiting set in phase space to which all trajectories are drawn as time tends to infinity, we say that the system dynamics are described by an attractor. The attractor is the geometrical limiting set on which all the trajectories eventually find themselves, that is, the set of points in phase space to which all the trajectories are attracted. After an initial transient period that depends on the initial state the orbit blends with the attractor. Attractors come in many shapes and sizes, but they all have the property of occupying a finite volume of phase space. As the system evolves it sweeps through the attractor, going through some regions rather rapidly and through others quite slowly, but always staying on the attractor after the initial transient. A discussion of these properties in a biomedical context is given by West (1990).

Whether or not the system is chaotic is determined by how two initially nearby trajectories cover the attractor over time. As Poincaré stated, a small change in the initial separation of any two trajectories may produce an enormous change in their final separation (sensitive dependence on initial conditions) — when this occurs the attractor is said to be "strange." The two nearby orbits separate exponentially in time which is to say the distance between them increases geometrically in each unit of time. The question is how this exponential separation is accomplished on a strange attractor of finite size. The answer to this question has to do with the layered structure necessary for an attractor to be chaotic. The transverse cross section of the layered structure of a strange attractor is a fractal. This property of a strange attractor relates the geometry of fractals to the dynamics of chaos.

Rössler (1976) described chaos as resulting from the geometrical operations of stretching and folding, often exemplified by the Baker's transformation. In this analogy the baker takes some dough and rolls it out on a floured bread board. When the rolled dough is thin enough the baker folds it back onto itself and rolls it out again. The map that carries out this operation is the Baker's transform. This process of rolling and folding is repeated again and again. Arnold gave a memorable image of this process using the image of the head of a cat [compare Arnold and Avez (1968)]. In Figure 6 a cross section of the square of dough is shown with the head of a cat inscribed. After the first rolling operation the head is flattened and stretched, that is, it becomes half its height and twice its length, as shown to the right at the top of the figure. It is then cut in the center and the segment of dough to the right is set above the one on the left to reform the initial square. The operation is repeated again and we see that at the bottom the cat's head is now

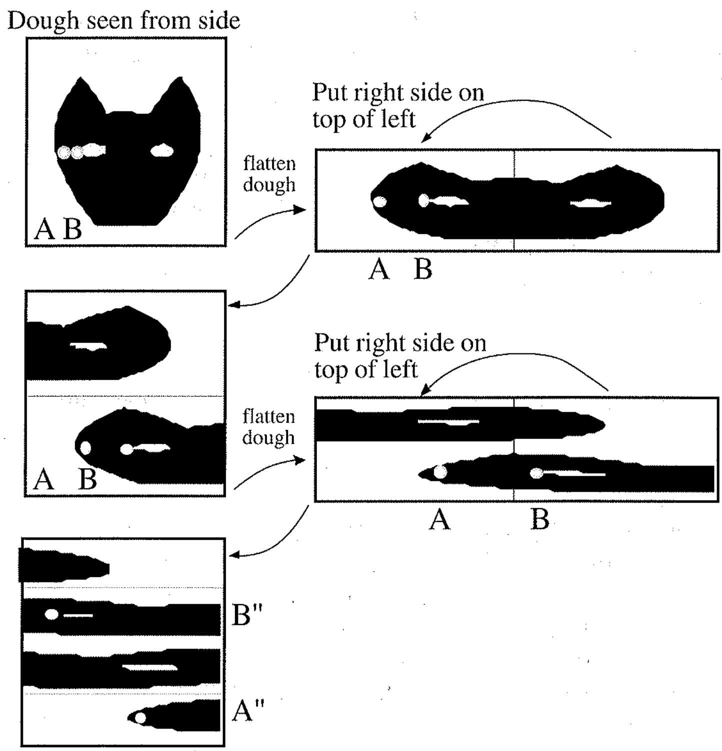

Figure 6: Arnold's cat being decimated by the stretching and folding operation that accompanies the dynamics of a chaotic attractor. After only two operations the cat's head is unrecognizable. Compare with *Ergodic Problems in Classical Mechanics* (p. 9), by V.I. Arnold and A. Avez, 1968, New York: Benjamins.

embedded in four layers of dough. Even after two of these transformations the cat's head is clearly decimated. After twenty stages of transformation the head will be distributed across a million layers of dough — not easy to identify. As so charmingly put by Ekeland (1988): "Arnold's cat has melted into the square, gradually disappearing from sight like the Cheshire Cat in Wonderland" (p. 51). The above argument of Rössler turned out to be generic. Two initially nearby orbits cannot rapidly separate forever on a finite attractor, therefore the attractor must eventually fold over onto itself. Once folded the attractor is again stretched and folded again. This process is repeated over and over, yielding an attractor structure with an infinite number of layers to be traversed by various trajectories. The infinite richness of the attractor structure affords ample opportunity for trajectories to diverge and follow increasingly different paths. The finite size of the attractor insures that these diverging trajectories will eventually pass close to one another again, albeit on different layers of the attractor. One can visualize these orbits on a chaotic attractor as being shuffled by this process, much as a deck of cards is shuffled by a dealer. Thus the randomness of the chaotic orbits is a consequence of this shuffling process. This process of stretching and folding creates folds within folds ad infinitum, resulting in the attractor having a fractal structure in phase space. The essential fractal feature of interest here is that the greater the magnification of a region of the attractor, the greater the degree of detail that is revealed, that is, one observes that there is no characteristic scale for the attractor and therefore none in the resulting time series given by a trajectory traversing the attractor. It should be emphasized that although this mathematical definition of an attractor is rigorous, it may not be the appropriate descriptor of the phenomenon of interest. For example, such a concept would only be approximate in dealing with certain aspects of the brain which is neither an autonomous nor stationary system. Such real phenomena would require generalizations of the notions of trajectories, attractors and evolution (Ott, 1993). The fourth non-traditional truth, then is: *nonlinear, deterministic equations of motion whether discrete or continuous do not necessarily have predictable final states due to the sensitivity of the solutions to initial conditions*.

Principle of Superposition

A principle that perhaps falls just short of a fundamental truth is that of superposition, but even so it is sufficiently pervasive that we treat it as the fifth traditional truth. It is worthwhile to note that the principle of superposition not only influenced development in the physical sciences and later dominated the thinking of scientists studying natural phenomena, but it also influenced our interpretation of the behavior of society and the nature of life itself. The principle can be stated as follows: *a complex process can be decomposed into constituent elements, each element can be studied individually and reassembled to understand the whole*. In this view of the world, all phenomena can be understood by treating the effects of nonlinearities as perturbations, that is, nonlinearities are always assumed to be *weak* effects. This would also suggest that the dominant behavior of a system is determined by its linear behavior and that the nonlinearity can make quantitative but not qualitative changes in the system's evolution. This was certainly the assumption of nearly all of nineteenth century physics. In this view the whole can never be more than the sum of its parts. There can be no emergent properties. If a property is not contained in the linear elements, reductionistically obtained, it cannot emerge from their superposition. The term *microreductionistic* is used by Causey (1969) to denote "an explanation of the behavior of a structured whole in terms of the laws governing the parts of the whole" (p. 230). Without denigrating the obvious success of this dictum in science and in our technological society we must also face its limitations when dealing with complex phenomena. The existence of chaos argues against the universal application of superposition and the microreductionistic ideas. Kellert (1993) contends that this does not argue against the validity of the philosophical doctrine of reductionism, however, which states that all properties of a system are reducible to the properties of its parts. He states: "Chaos theory gives no example of 'holistic' properties which could serve as counterexamples to such a claim" (p. 90). A segment of the scientific community disagrees with this assertion, so let me point out that the statistical distribution resulting from a chaotic time series is a property of the system as a whole and cannot be traced back to any of its isolated constituent parts. For example, it is well known that two bodies acting under a mutual gravitational force give rise to a solvable set of equations using Newton's laws. However, three bodies interacting under the same force yield a set of equations that are not solvable in the traditional sense and indeed yield chaotic solutions. Thus a complex (deterministic, nonlinear and dynamical) system is irreducible and is chaotic in general. Chaos itself is a consequence of the stability of the entire system and the structure of the attractor upon which the system dynamics unfold, not of any component part or collection or parts, so I would maintain that chaos undercuts the foundations of the philosophical doctrine of reductionism. Understanding the process as a whole cannot be achieved through a knowledge of the decomposed elements. The evolution of such systems cannot be described by perturbation theory. Therefore the fifth non-traditional truth is that *most complex phenomena do not lend themselves to the principle of superposition*.

Summary

Table 1 summarizes the change in perspective that has been developing in the physical sciences in the past two decades, and is more compatible with the view of the world the life scientists and psychologists have held all along. The cool, predictable, almost sterile view of natural phenomena held by the classical physical scientists is being overwhelmed in the acceptance of the unpredictable, ever-changing phenomena found in the social and psychological sciences. As the old and new views coalesce the modern mathematical tools help to make understandable the irregular and erratic features of everyday life.

The Value of Chaos in the Social and Life Sciences

A final question with which to end this overview of chaos is: What is the value of chaos in the social and psychological sciences? Conrad (1986) has suggested five functional roles that chaos might play in biological systems: (a) search, (b) defense, (c) maintenance, (d) cross-level effects and (e) dissipation of disturbances. Each of these roles is found to have a sociological or psychological analog.

In the first function, that of search, chaos acts to enhance exploratory activity, independently of whether one is dealing with predators searching for prey, the brain devising strategies for accessing memory, or a individual seeking gratification in a social context.

In the second function, that of defense, the diversity of behavior is used to avoid predators rather than to explore the environment. An organism that moves about in an unpredictable way is certainly more difficult to ensnare than one that moves regularly in circles or along a straight line. This may also apply to the speeches of politicians.

A third possible function for chaos is the prevention of entrainment, which is to say, the maintaining of the process. It has been argued here and elsewhere that a complex system whose individual elements act more independently are more adaptable than one in which the separate elements are tightly locked.

A fourth function, that of cross-level effects, has to do with micro and macroscopic effects. The micro is at the level of the interaction of individuals and the macro is at the level of societal forces analogous to Darwin's mechanism of variation and selection. The variability due to the creativity of the individual may well lead to evolution of new social entities, for example, books, public libraries and computers to name a few.

Dissipation of disturbances is a fifth possible function of chaos. This has been known for some time in the physical sciences (see, for example, Trefán, Grigolini, and, West [1992]). If the erratic behavior of a social system is produced by a strange attractor on which all trajectories are functionally equivalent, the sensitivity to the initial conditions is the most effective mechanism for dissipating disturbances, since the disturbance is so soon mixed with previously existing orbits. Thus labor strikes and other conflicts are absorbed by the erratic unfolding of social progress.

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